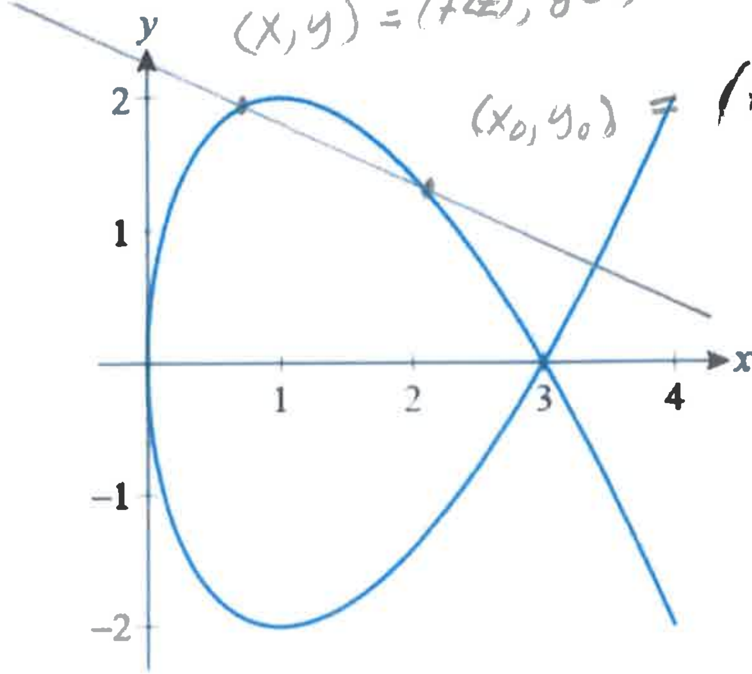
Calculus and Parametric Equations



$(x, y) = (f(t), g(t))$

$(x_0, y_0) = (f(t_0), g(t_0))$

Definition:

A curve C defined by the parametric equations $x = f(t)$ *and* $y = g(t)$ is differentiable at t_0 if both f and g are differentiable at t_0 . The parametrization is smooth at t_0 if both f' and g' are continuous and at least one of $f'(t_0)$ and $g'(t_0)$ is nonzero.

What is the slope of the line tangent to the curve at $(x_0, y_0) = (f(t_0), g(t_0))$?

$$= \lim_{t \to t_0} \frac{y - y_0}{x - x_0} = \lim_{t \to t_0} \frac{g(t) - g(t_0)}{f(t) - f(t_0)} =$$

$$\lim_{t \to t_0} \frac{\frac{g(t) - g(t_0)}{t - t_0}}{\frac{f(t) - f(t_0)}{t - t_0}} = \frac{g'(t_0)}{f'(t_0)}$$

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$

Formula: Parametric Formula for $\frac{dy}{dx}$

Assume the curve C is defined parametrically by $x = f(t)$ and $y = g(t)$. If each of the three derivatives $\frac{dy}{dx}$, $\frac{dy}{dt}$, and $\frac{dx}{dt}$ exists at a given point, and if $\frac{dx}{dt} \neq 0$, then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} =$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

The curve C has a vertical tangent line (and hence $\frac{dy}{dx}$ does not exist) at a point of smoothness where $\frac{dx}{dt} = 0$ (that is, $\frac{dy}{dt} \neq 0$ and $\frac{dx}{dt} = 0$).

Example:

Find the equation of any horizontal or vertical tangent line to the curve $x = t^2 - t$, $y = 1 + 6t^2$.

$f(t) = t^2 - t$, $g(t) = 1 + 6t^2$

$f'(x) = 2t - 1$, $g'(t) = 12t$

At $t = 0$, $g'(0) = 0$ and $f'(0) = -1 \neq 0$,

so there is a horizontal tangent line

at $(f(0), g(0)) = (0, 1)$.

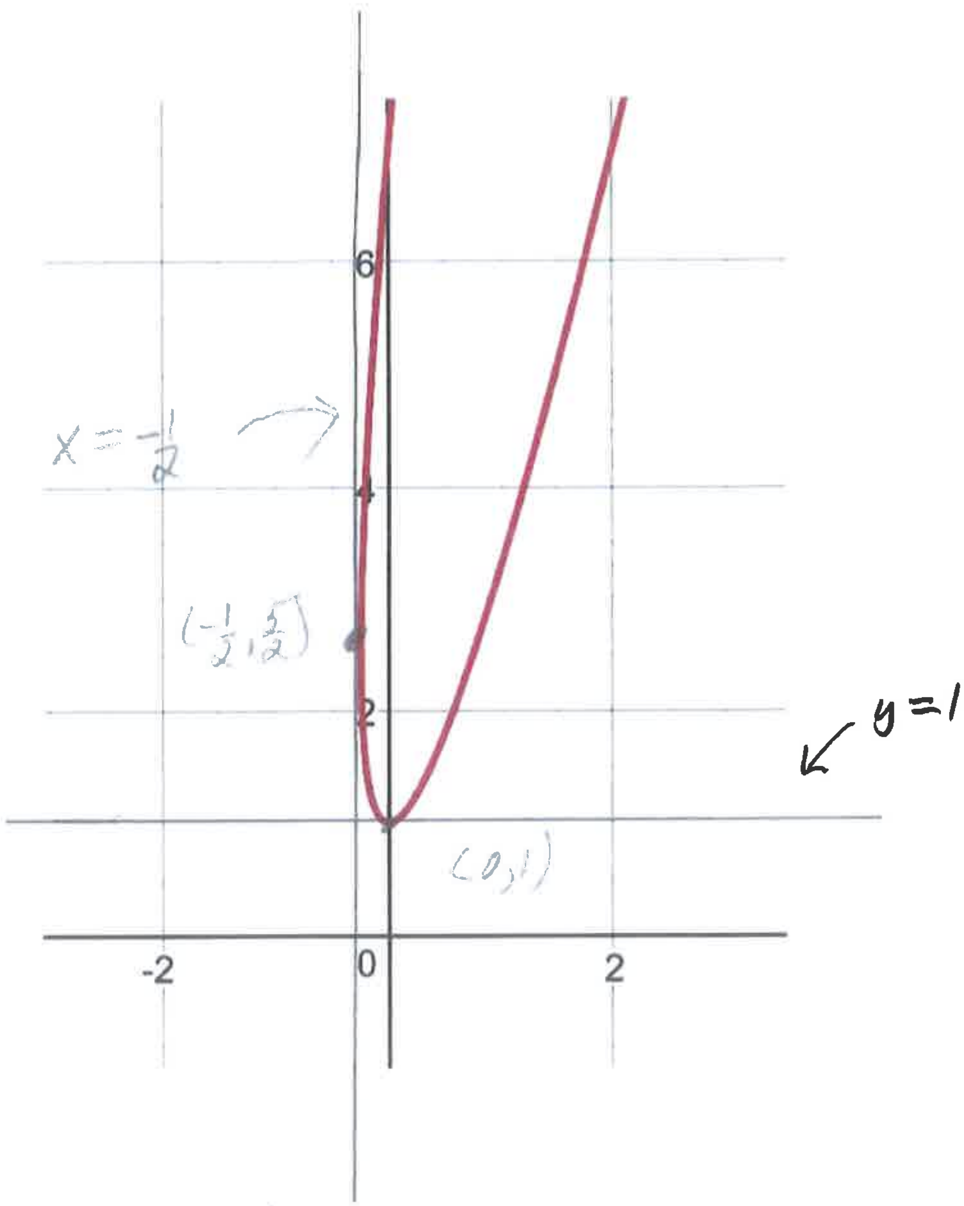
The equation is $y = 1$

At $t = \frac{1}{2}$, $f'(\frac{1}{2}) = 0$ and $g'(\frac{1}{2}) = 6 \neq 0$,

so there is a vertical tangent line

at $(f(\frac{1}{2}), g(\frac{1}{2})) = (-\frac{1}{2}, \frac{5}{2})$

The equation is $x = -\frac{1}{2}$.



6
4
2
-2
0
2
$x = -\frac{1}{2}$
$(-\frac{1}{2}, \frac{5}{2})$
$y = 1$
$(0, 1)$